



TITLE:

TOPOLOGY OF THE SPACE ON WHICH CELLULAR AUTOMATA WORKS (Algebraic System, Logic, Language and Related Areas in Computer Science)

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TOPOLOGY OF THE SPACE ON WHICH CELLULAR AUTOMATA WORKS

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ABSTRACT. We investigate :

- (A) Topology of the product of discrete topological spaces
and
- (B) Homeomorphism of two topologies (T_p) and (T_m) of cell
space $C = S^{\mathbb{Z}^n}$ of a cellular automaton $\mathcal{A} = (\mathbb{Z}, S, N, f)$.

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1. PRODUCT OF DISCRETE TOPOLPGICAL SPACES

Definition.

- X_i : discrete topological space for i in I
 \mathcal{O}_i : the system of open sets of X_i ,
- $X = \prod_{i \in I} X_i$: the product of X_i of weak topology
 \mathcal{O} : the system of open sets of X ,
- and so
- $\mathcal{O}_0 = \{\prod_{j \in J} O_j \times \prod_{i \in I \setminus J} X_i \mid O_j \in \mathcal{O}_j, J \subseteq I, |J| < \infty\}$
: an open basis of \mathcal{O} .

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Now we state our theorem.

Theorem A. For X_i 's discrete the following are equivalent:

- (a) $X = \prod_{i \in I} X_i$ is discrete.
- (b) $|\{i \in I \mid |X_i| \geq 2\}| < \infty$,
that is, $|X_i| = 1$ for almost all i in I .

Corollary. The product of discrete topological spaces is discrete if and only if it is homeomorphic to a product of a finite number of discrete topological spaces. In particular, if S is a discrete topological space containing at least two elements, then its infinite product is not discrete.

Proof.

- (a) $\Rightarrow X \ni \forall x : \text{open,}$ (since X is discrete)
 $\Rightarrow X \ni \forall x = \cup_{\lambda \in \Lambda} O_\lambda$ for some O_λ 's $\in \mathcal{O}_0$,
 (for \mathcal{O}_0 is an open basis of \mathcal{O})
 $= O_\lambda$ for any λ in Λ ,
 (since $|\{x\}| = 1$)
 \Rightarrow setting $O_\lambda = \prod_{j \in J} x_j \times \prod_{i \in I \setminus J} X_i$ for some J , $|J| < \infty$
 and $x = \prod_{j \in J} x_j \times \prod_{i \in I \setminus J} x_i$,
 we have
 $\Rightarrow X_i = x_i$ for i in $I \setminus J$ with $|J| < \infty$,
 \Rightarrow (b).

- (b) \Rightarrow For

$$J = \{i \in I \mid |X_i| \geq 2\}$$

we have

$$|J| < \infty$$

\Rightarrow Choose

$$X \ni \forall x = \prod_{j \in J} x_j \times \prod_{i \in I \setminus J} x_i, \quad x_i \in X_i, \quad x_j \in X_j,$$

Then by the definition of J

$$\forall i \in I \setminus J, \quad x_i = X_i$$

and so

$$X \ni \forall x = \prod_{j \in J} x_j \times \prod_{i \in I \setminus J} X_i \in \mathcal{O}_0, \quad x_j \in X_j$$

\Rightarrow (a)

Q.E.D.

2. THE TOPOLOGY OF THE CONFIGURATION SET C OF A CELLULAR AUTOMATON \mathcal{A}

◦ $\mathcal{A} = (\mathbb{Z}, S, N, f)$: a cellular automaton

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\} : \text{rational integers,}$$

◦ $\mathbb{Z}^n = \mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}$: cell space

◦ $S = \{s_1, s_2, \dots, s_m\}$: states

◦ $f : S^l \rightarrow S$: local map

Further we define

◦ $C = S^{\mathbb{Z}^n} = \text{Map}(\mathbb{Z}^n, S)$: configurations

where we give

◦ S : the discrete topology

Then $C = S^{\mathbb{Z}^n}$ has two topologies (T_p) and (T_m) following :

(T_m)

For

$$\forall c \in C = S^{\mathbb{Z}^n},$$

$$\forall \epsilon = 2^{-\lambda}, \quad \lambda \in \mathbb{N}$$

define

$$U_\epsilon(c) = \{c' \in C \mid d(c, c') < \epsilon\}$$

: the ϵ -neighbourhood of c

where

$$d(c, c') = 2^{-\min\{\delta(0, i) \mid c(i) \neq c'(i), i \in I\}} \text{ for } c, c' \text{ in } C,$$

$$\delta(0, i) = \sqrt{i_1^2 + i_2^2 + \dots + i_n^2} \text{ for } i = (i_1, i_2, \dots, i_n) \text{ in } \mathbb{Z}^n.$$

(**T_p**)

For

$$\forall c \in C = S^{\mathbb{Z}^n},$$

$$\forall J \subseteq \mathbb{Z}^n, \quad |J| < \infty$$

define

$$V_J(c) = (\prod_{j \in J} c(j)) \times S^{\mathbb{Z}^n \setminus J}$$

: the J -neighbourhood of c

Then, since S is discrete, we have $c_j \in \mathcal{O}_j$ for any $j \in J$. This enable us to take

$$\{V_J(c) \mid c \in C, J \subseteq \mathbb{Z}^n, \quad |J| < \infty\}$$

as \mathcal{O}_0 an open basis.

Then, we have :

Theorem B.

$$(\mathbf{T}_p) \simeq (\mathbf{T}_m) : \text{ homeomorphic}$$

Proof. (a) We show $(\mathbf{T}_p) \leq (\mathbf{T}_m)$, i.e., $\exists U_\epsilon(c) \subseteq \forall V_J(c)$

For

$$\forall V_J(c) = \prod_{j \in J} c(j) \times S^{\mathbb{Z}^n \setminus J}$$

with

$$\forall c \in C = S^{\mathbb{Z}^n},$$

$$\forall J \subseteq \mathbb{Z}^n \text{ and } |J| < \infty$$

choose $\lambda \in \mathbb{N}$

$$\text{such that } (\mathbf{A}) \quad \forall j \in J, \quad \delta(0, j) < \lambda$$

and set

$$\forall \epsilon = 2^{-\lambda}$$

Then,

$$\begin{aligned}
c' \in U_\epsilon(c) &\Rightarrow d(c, c') < \epsilon = 2^{-\lambda}, \text{ where} \\
d(c, c') &= 2^{-\min\{\delta(0, i) \mid c(i) \neq c'(i)\}} \text{ for } c, c' \text{ in } C, \\
&\Rightarrow \lambda < \min\{\delta(0, i) \mid c(i) \neq c'(i)\} \\
&\Rightarrow \text{By (A)} \\
&\quad \text{if } j \in J, \text{ we have } \delta(0, j) < \lambda \\
&\quad \text{and so } c(j) = c'(j) \\
&\Rightarrow c' \in V_J(c) = \prod_{j \in J} c(j) \times S^{\mathbb{Z}^n \setminus J}
\end{aligned}$$

Thus,

$$U_\epsilon(c) \subseteq V_J(c) \text{ and so } (T_p) \leq (T_m).$$

(b) Next we show $(T_m) \leq (T_p)$, i.e., $\exists V_J(c) \subseteq {}^\forall U_\epsilon(c)$.

For

$$\begin{aligned}
&{}^\forall U_\epsilon(c) \\
&\quad \text{with} \\
&\quad c \in C = S^{\mathbb{Z}^n}, \\
&\quad \epsilon = 2^{-\lambda}, \quad \lambda \in \mathbb{N}
\end{aligned}$$

let

$$(B) \quad J = \{j \in I \mid \delta(0, j) \leq \lambda\}$$

Then, since

$$V_J(c) = \prod_{j \in J} c(j) \times S^{\mathbb{Z}^n \setminus J},$$

we have

$$\begin{aligned}
c' \in V_J(c) &\Rightarrow {}^\forall j \in J, \quad c(j) = c'(j) \\
&\Rightarrow \text{by (B) if } \delta(0, j) \leq \lambda, \text{ we have } j \in J \\
&\quad \text{and so } c(j) = c'(j) \\
&\Rightarrow \lambda < \min\{\delta(0, j) \mid c(j) \neq c'(j)\} \\
&\Rightarrow 2^{-\min\{\delta(0, j) \mid c(j) \neq c'(j)\}} < 2^{-\lambda} = \epsilon \\
&\Rightarrow d(c, c') < \epsilon \\
&\Rightarrow c' \in U_\epsilon(c)
\end{aligned}$$

Thus

$$V_J(c) \subseteq U_\epsilon(c) \text{ and so } (T_m) \leq (T_p).$$

Q.E.D.

REFERENCES

- [1] B. Durand, Global Properties of Cellular automata, (E. Goles, S. Martinez, Eds. , Cellular automata Complex Systems, Kluwer Academic Publishers), (1999) 1-22.
- [2] H. Ishibashi, Injectivity of Cellular Automata, Jp Journal of Algebra, Number Theory and Appl. 40(2) (2018) , 199-205.
- [3] H. Ishibashi, Product of Discrete Topological Spaces, Jp Journal of Algebra, Number Theory and Appl. 40(5) (2018) , 685-690.
- [4] H. Nishio, T. Saito, Information of Cellular Automata I - An Algebraic Study -, Fundamenta Informaticae 58 (2003) 399-420.

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